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SHELLS OF REVOLUTION FREE OF BENDING UNDER UNIFORM AXIAL LOADING

R. CHICUREL

National Autonomous University of Mexico, México, D.F., México

Abstract—The meridian geometry of a shell of revolution may be chosen so that, if clamped or simply supported end conditions exist, neither bending moments nor transverse shear forces will occur anywhere when uniform axial load is applied. For an isotropic shell, an exact solution for the shell configuration satisfying this condition is presented. Because of the importance of transverse shear as a cause of delamination in composite materials, consideration is given to the case of a fiber reinforced shell.

NOTATION

A, B, C	constants
a	radius of end cross-section of shell
E	modulus of elasticity
h	wall thickness
N_ϕ	in-plane meridional normal force/unit length
N_θ	in-plane circumferential force/unit length
q	axial compressive force per unit circumferential length of end cross-section
r	radius of transverse cross-section of shell
r_1	principal radius of curvature in meridional plane
r_2	principal radius of curvature normal to meridional plane
v	displacement component in meridional direction.
w	displacement component in normal direction
y	axial distance from mid-section
η	transverse component of displacement
ν	Poisson's ratio
ϕ	angle between normal to middle surface and shell's axis
$(\,)'$	$d/d\phi$

INTRODUCTION

THE condition for suppressing bending and transverse shear in a thin shell of revolution loaded axisymmetrically is that the meridional and normal components of displacement satisfy the relation

$$v \equiv -w'. \tag{1}$$

Equation (1) was given by Murthy [1] and Murthy and Kiusalaas [2] for isotropic shells. Pao [3] and Chicurel and Wu [4] observed that the condition is also applicable in the case of orthotropic shells, where the axes of orthotropy coincide with the meridional and circumferential directions and where it is tacitly assumed that no coupling between bending and extension exist in the constitutive equations.

The appropriate equilibrium and stress–displacement equations are:

$$\text{equilibrium}\begin{cases}(N_\phi r)' - N_\theta r_1 \cos\phi = 0 & (2)\\ N_\phi r + N_\theta r_1 \sin\phi = 0 & (3)\end{cases}$$

$$\text{stress–displacement}\begin{cases}N_\phi = -\dfrac{E_\phi h}{1-\nu_\theta\nu_\phi}\left[\dfrac{1}{r_1}(v'-w)+\dfrac{\nu_\theta}{r_2}(v\cot\phi - w)\right] & (4)\\[2ex] N_\theta = \dfrac{E_\theta h}{1-\nu_\theta\nu_\phi}\left[\dfrac{1}{r_2}(v\cot\phi - w)+\dfrac{\nu_\phi}{r_1}(v'-w)\right]. & (5)\end{cases}$$

From the solution of (2) and (3), we have

$$N_\phi = \frac{C_1}{r\sin\phi}, \qquad N_\theta = -\frac{C_1}{r_1\sin^2\phi},$$

where C_1 is an integration constant. If these expressions are substituted in equations (4), (5) and if the transverse displacement $\eta = w\sin\phi - v\cos\phi$, together with the bending suppression condition (1) is used, two equations with unknowns η, r, r_1, r_2 will result. However, r_1 and r_2 are easily eliminated by applying the relations

$$r_1 = r'/\cos\phi, \qquad r_2 = r/\sin\phi.$$

Thus, a pair of differential equations for η, r will result. These are:

$$\eta' r + \nu_\theta \eta r' = -\frac{(1-\nu_\phi\nu_\theta)C_1 r'}{E_\phi h\sin\phi} \tag{6}$$

$$\nu_\phi \eta' r + \eta r' = \frac{(1-\nu_\phi\nu_\theta)C_1 r\cos\phi}{E_\theta h\sin^2\phi}. \tag{7}$$

ISOTROPIC SHELL UNDER AXIAL LOAD

The special case of an isotropic shell under uniform axial tension or compression loading will now be considered. The end conditions will be taken to correspond to simple supports. However, it should be noted that, since the condition expressed by equation (1) is equivalent to specifying that the rotation in the meridional plane vanish, there is no distinction between a simply supported and a clamped edge, so that both conditions are satisfied simultaneously.

For the isotropic shell, we put $\nu_\phi = \nu_\theta = \nu$, $E_\phi = E_\theta = E$. If then, equations (6) and (7) are added, the resulting equation may be written as

$$(1+\nu)(\eta r)' = -\frac{(1-\nu^2)C_1}{Eh}\left(\frac{r}{\sin\phi}\right)'$$

and, in integrated form,

$$\eta r = -\frac{(1-\nu)C_1 r}{Eh\sin\phi} + C_2. \tag{8}$$

If equation (6) is multiplied by ν and is then subtracted from (7), we obtain, after dividing by r',

$$\eta = \frac{C_1}{Ehr'\sin\phi}(r\cot\phi + \nu r'). \tag{9}$$

The expression for η given by equation (9) may be inserted in equation (8) to arrive at a single differential equation for r which, after some manipulation, may be put in the form

$$\frac{C_1}{Eh}\left(\frac{1}{r\sin\phi}\right)' - C_2\left(\frac{1}{2r^2}\right)' = 0. \tag{10}$$

Integrating equation (10) produces a quadratic equation in r, whose solution is:

$$r = \frac{1}{2}\left(\frac{C_1}{C_3 Eh\sin\phi} \pm \sqrt{\left(\frac{C_1}{C_3 Eh\sin\phi}\right)^2 - \frac{2C_2}{C_3}}\right), \tag{11}$$

where another constant of integration, C_3, has been introduced.

At an end, we specify $r = a$, $\phi = \phi_0$ and $\eta = 0$. Applying these conditions to equations (8) and (11) permits evaluation of the constants C_1 and C_2 in terms of C_3. In this manner, equation (11) becomes, after some rearrangement:

$$r = \frac{a}{(1+v)\sin\phi}\left\{\sin\phi_0 \pm \sqrt{\sin^2\phi_0 - (1-v^2)\sin^2\phi}\right\}. \tag{12}$$

Substituting $\phi = \phi_0$ in equation (12) will reveal that $r = a$ will result only if a plus sign before the radicand is accepted. Therefore, the minus sign will be ignored hereafter.

It may easily be shown that the only values of ϕ which yield $\eta = 0$ are ϕ_0 and $[(\pi/2)-\phi_0]$. In either case, $r = a$. Noting also that the expression for r given by equation (12) is an even function of ϕ about $\phi = \pi/2$, it is apparent that the only possible configuration is one with a plane of symmetry about the cross-section $\phi = \pi/2$.

In order to avoid complex values of r, it is clear from equation (12) that the minimum value that $\sin^2\phi_0$ may have is $(1-v^2)$, or:

$$\cos\phi_0 \leq v. \tag{13}$$

If $v = \frac{1}{3}$, ϕ_0 cannot be less than 70·55°.

Now let y represent the axial distance from the mid-section, measured positively in the direction in which ϕ decreases. Then:

$$y' = r_1\sin\phi = r'\tan\phi. \tag{14}$$

Substituting for r in equation (14) by use of equation (12) produces the following expression for y':

$$y' = \frac{-a\sin\phi_0}{(1+v)\sin\phi}\left\{1 + \frac{\sin\phi_0}{\sqrt{\sin^2\phi_0 - (1-v^2)\sin^2\phi}}\right\}$$

which may be integrated to give:

$$y = \frac{a\sin\phi_0}{(1+v)}\left\{-\ln\left(\tan\frac{\phi}{2}\right) + \sinh^{-1}\left(\frac{\sin\phi_0\cot\phi}{\sqrt{v^2-\cos^2\phi_0}}\right)\right\}. \tag{15}$$

Equations (12) and (15) represent parametric equations of the meridian curve where r and y may be considered cartesian coordinates and ϕ a parameter. For each value of ϕ_0 chosen, a shape will be obtained. Figure 1 exhibits some of these shapes. The end radius, a, was kept constant for all values of ϕ_0.

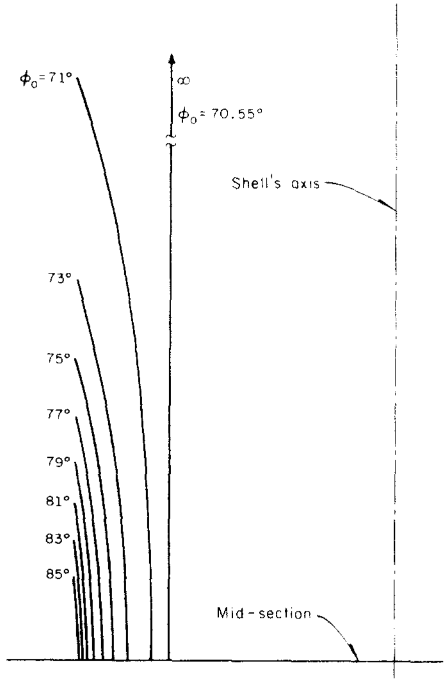

FIG. 1. Meridian geometry of axially loaded shells of revolution for bending suppression, $\nu = \frac{1}{3}$.

An expression for the meridional radius of curvature may be found by integrating $r_1 = r'/\cos\phi$ after r is put in terms of ϕ. The result is:

$$r_1 = \frac{-a\sin\phi_0}{(1+\nu)\sin^2\phi}\left\{1+\frac{\sin\phi_0}{\sqrt{\sin^2\phi_0-(1-\nu^2)\sin^2\phi}}\right\}. \tag{16}$$

If q represents the intensity of uniform axial compression applied at the ends of the shell, then at $\phi = \phi_0$ we have

$$N_\phi \sin\phi_0 = q. \tag{17}$$

From the expressions for N_ϕ, N_θ written after equation (5) and from equations (12), (16) and (17) one can arrive at the following equations expressing the meridional variation of N_ϕ, N_θ:

$$N_\phi = \frac{-(1+\nu)q}{\sin\phi_0+\sqrt{\sin^2\phi_0-(1-\nu^2)\sin^2\phi}} \tag{18}$$

$$N_\theta = \frac{-(1+\nu)q}{\sin\phi_0\left[1+\frac{\sin\phi_0}{\sqrt{\sin^2\phi_0-(1-\nu^2)\sin^2\phi}}\right]}. \tag{19}$$

FIBER REINFORCED AXIAL LOAD MEMBER

Since transverse shear is absent in the configurations considered in this study, the discussion is of particular interest in regards to composite materials susceptible to delamination [3, 4]. This motivated the examination of a fiber reinforced shell with alternating layers of axially and circumferentially oriented fibers. The density of the axial fibers would vary inversely with r, while it is assumed that the density of circumferential fibers is constant. If, then, the Halpin–Tsai equations [5] for the equivalent elastic constants for the laminate are employed, it is not difficult to show that E_ϕ, E_θ, ν_ϕ would be functions of r of the forms:

$$E_\phi = A_1 + B_1/r \tag{20}$$

$$E_\theta = A_2 + B_2\left(\frac{r+2C}{r-C}\right) \tag{21}$$

$$\nu_\phi = A_3 + B_3/r \tag{22}$$

$$\nu_\theta = E_\theta \nu_\phi / E_\phi \tag{23}$$

where the A's, B's and C are constants. Substitution of the expressions given by equations (20–(23) into equations (6) and (7) results in a set of differential equations for r, η which could be integrated numerically to yield the bending free geometry.

CONCLUDING REMARKS

Figure 1 shows that, by varying ϕ_0, a shell of any length to end diameter ratio may be obtained. Thus, long members could be used to form bar-type structures. It is to be noted that the mid-section diameter has a finite limit as the length approaches infinity.

For an intuitive grasp of the problem examined in this study, it is helpful to imagine a thin rubber membrane which is initially cylindrical and has rigid disks attached to the ends. If it is stretched by pulling the ends apart, a configuration similar to the curves of Fig. 1 will result. In fact, if the initial proportions of the cylinder and the amount of stretch is properly chosen, one could obtain a configuration as predicted by the present formulation. A verification of the theoretical expressions for N_ϕ, N_θ and η for the isotropic shell has been made by numerical integration [6]. Some of the results are shown in Table 1.

Numerical results for orthotropic shells under axial compression plus internal pressure have been presented in an extension of the present study [7]. Another related study being

TABLE 1. RESULTS FOR SHELL WITH $a = 5{\cdot}3282$ in., $h = 0{\cdot}020$ in., $\phi_0 = 75°$, $\nu = 0{\cdot}33$, $E = 10^7$ psi, $q = 100$ lb/in.

ϕ (deg.)	η (in. × 10^{-4}) Analytical solution	η (in. × 10^{-4}) Numerical solution	N_ϕ (lb/in.) Analytical solution	N_ϕ (lb/in.) Numerical solution
75	0	0	−103·57	−103·53
80	−1·559	−1·559	−108·29	−108·19
85	−2·670	−2·668	−112·02	−111·88
90	−3·087	−3·086	−113·45	−113·30

carried out consists of theoretical and experimental evaluations of buckling loads for these shapes for the purpose of assessing the effect of suppressing pre-buckling bending on buckling strength. Studies such as that of Almroth [8] have established the fact that shell buckling analyses based on assuming a membrane pre-buckling state overestimate the buckling load as compared with more exact calculations which allow a bending pre-buckling condition. However, the configurations considered in the present investigation would, in fact, be free of bending in a pre-buckling condition. Hence, the buckling loads predicted by membrane pre-buckling analyses would not, in this case, include such over-estimation.

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Абстракт—Можно так выбрать геометрию Меридиана оболочек вращения для краевых условии прлного защемления и свободного опирания, что не будут появлятся нигде ни моменты изгиба, ни поперечные силы сдвига, для случая приложенной равномерной осевой нагрузки. Дается точное решение для изотропной оболочки, которое удовлетворяет условию конфигурации оболочки. Дается внимание для оболочек, усиленных волокнами, вследствие важности поиерелных сдвиго в кйк ирилцпы рчссицелця ь сиоисмых мймерицх.